Determination of Principal Axes and Principal Mass Moments of Inertia when no Planes of Symmetry Exist

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Abstract

The purpose of this article is to offer a deeper perspective towards a topic, hastily considered trivial by many engineers. The determination of principal axes and mass moments of inertia is regarded as an eigenvector and eigenvalue problem. Some theoretical background is provided for the case when no axes or planes of symmetry exist. The practical application is emphasized via example. Manual calculation is conducted and a MATLAB code is provided as well.

Keywords: Principal Axes of Inertia, Principal Mass Moments of Inertia, Eigenvectors, Eigenvalues

Simetri Düzlemleri Yokken Asal Eksenlerin ve Asal Kütle Eylemsizlik Momentlerinin Belirlenmesi

Öz

Bu makalenin amacı, birçok mühendis tarafından aceleyle önemsiz kabul edilen bir konuya daha derin bir bakış açısı sunmaktır. Asal eksenlerin ve kütle atalet momentlerinin belirlenmesi, bir özvektör ve özdeğer problemi olarak kabul edilir. Herhangi bir eksen veya simetri düzleminin olmadığı durum için bazı teorik altyapı sağlanmıştır. Pratik uygulama örnek üzerinden vurgulanmıştır. Manuel hesaplama yapılır ve bir MATLAB kodu da sağlanır.

Anahtar Kelimeler: Eylemsizliğin Ana Eksenleri, Eylemsizliğin Temel Kütle Momentleri, Özvektörler, Özdeğerler

1. Introduction

In a variety of engineering problems the need of determining the inertial characteristics emerges [1]. Design of cross sections, assemblance of machine part and optimal usage of material are examples of such problems. They all require the prediction of the behavior of objects or systems. Therefore, moments of

inertia need to be obtained. While this is a simple task when the body we are interested in has planes of material symmetry, the task becomes more complicated when no such planes exist [2]. Examples of such problems include not only geometrically asymmetric bodies, but bodies with unevenly distributed mass as well.

Cases of symmetric 2D bodies (e.g. rods, disks, rings) are covered in every course of mechanics that engineering students are required to take [3], [4]. However, problems involving 3D bodies with no planes or axes of symmetry do not usually get much attention in textbooks. This is easily explained with the fact that most of the elements (used in machines, building structures etc.) are produced as symmetric.

Solutions with experimental determination of inertial characteristics are known [5], but their accuracy depends on the precision with which the experiment is conducted, as well as on the accepted prerequisites. Moreover, nowadays the usage of analytical solutions that can be implemented numerically is preferred in engineering design.

However, a numerical solution cannot be realized without a proper understanding of the basic concepts which govern the calculation. The determination of moments and axes of inertia is not just a substitution in pre-derived formulas. Therefore, here we will show how to determine the principal axes and mass moments of inertia of a 3D body in general.

2. Material and Method

2.1. Theoretical Background

It is known that the problem of determining the principal axes and principal mass moments of inertia can be stated as an eigenvector and eigenvalue problem [2], [6].

Let us consider the definition of angular momentum $\vec{K}$ for a rigid body:

$$\vec{K} = \int_m (\vec{r} \times \vec{v}) dm \quad (1)$$

, where $\vec{r}$ is the radius vector (or the position vector) of a particle, $\vec{v}$ is the velocity vector of the particle and m is the mass of the particle.

Velocity can traditionally be written as a cross product of the angular velocity vector (denoted as $\vec{\omega}$ in the following equations) and the radius vector [7]. Then the linear algebra relation among cross and dot product of three vectors can be used [8]:

$$\vec{K} = \int_m (\vec{r} \times (\vec{\omega} \times \vec{r})) dm = \int_m [(\vec{r}\vec{r})\vec{\omega} - (\vec{r}\vec{\omega})\vec{r}] dm \quad (2)$$

, where for any three vectors a, b and c:

$$a \times (b \times c) = (ac)b - (ab)c \quad (3)$$

After some mathematical transformations one can obtain the result [2]:

$$\vec{K} = \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad (4)$$

, where I_x, I_y and I_z are axial mass moments of inertia and $I_{xy} = I_{yx}, I_{xz} = I_{zx}, I_{zy} = I_{yz}$ are products of inertia with respect to x, y and z axes of a Cartesian coordinate system.

It is also known that the angular momentum vector is collinear with the axis of rotation (I is the mass moment of inertia with respect to that axis), which is represented with the equation:

$$\vec{K} = I\vec{\omega} \quad (5)$$

If the angular momentum is to be described in system of principal axes of inertia, eq. (4) is written as:

$$\vec{K} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad (6)$$

Equation (7) is obtained by equating equations (4) and (6). Such operation is valid for the angular momentum is a vector that can be considered in different coordinate systems. A system of linear equations is derived that has solutions if:

$$\det \begin{vmatrix} I_x - I & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y - I & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z - I \end{vmatrix} = 0 \quad (7)$$

By solving the above determinant a cubic equation is given:

$$I^3 - AI^2 + BI - C = 0 \quad (8)$$

, where the coefficients A, B and C are invariant with respect to the coordinate system in which they are considered. These coefficients can be calculated as follows:

$$A = I_x + I_y + I_z \quad (9)$$

$$B = I_x I_y + I_y I_z + I_x I_z - (I_{xz})^2 - (I_{zy})^2 - (I_{yx})^2 \quad (10)$$

$$C = \det \begin{vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{vmatrix} \quad (11)$$

Equations (9), (10) and (11) can be used for checking the obtained eigenvalues. It is worth noting that eq.(8) is similar to Cauchy's equation of three-dimensional state of stress and eq. (9), (10), (11) are similar to the stress invariants [9]. The solutions of eq. (8) are the principal mass moments of inertia of the rigid body. The direction cosines λ , μ and ν of the principal axes of inertia can be determined from the system:

$$\begin{cases} \lambda(I_x - I_i) + \mu(-I_{xy}) + \nu(-I_{xz}) = 0 \\ \lambda(-I_{yx}) + \mu(I_y - I_i) + \nu(-I_{yz}) = 0 \\ \lambda(-I_{zx}) + \mu(-I_{zy}) + \nu(I_z - I_i) = 0 \\ \lambda^2 + \mu^2 + \nu^2 = 1 \end{cases} \quad (12)$$

System (12) is solved with 3 of its 4 equations as the last one (sum of the squares of the direction cosines equals 1) is mandatory to use. This is done three times – once per eigenvalue I_i .

In essence, this is a problem of determining eigenvalues (principle mass moments of inertia) and eigenvectors (principal axes of inertia) of a matrix (tensor of inertia). The matrix (13) is used to describe the rigid body:

$$\begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix} \quad (13)$$

3. Results and Discussion

3.1. Example

The example given below is of a body for which a plane of symmetry exists, for it is easier to comprehend the steps involved in the solution of the problem and verify the results. The same procedure holds true if we are considering a 3D body in general. The problem itself is stated with a Cartesian coordinate system as shown at fig. 1.

Figure 1. Geometry of a rigid body and orientation of the selected coordinate system

The body itself lies in plane Oxy, it is composed of three rods and three point masses at the ends of those rods. Length of rods is $L = 2 \text{ m}$. Mass per unit length is $m_l = 2 \text{ kg/m'}$, while the point masses are $m = 3 \text{ kg}$. Using the well-known formulas for mass moments of inertia with respect to rotated axes one can obtain matrix (tensor) of inertia (13) in Cartesian coordinate system as shown at fig. 1 [10].

$$I_x = I_y = \frac{1}{3} \cdot (2 \cdot 2) \cdot 2^2 + 3 \cdot 2^2 + 3 \cdot 1,414^2 + \frac{1}{3} \cdot (2 \cdot 2) \cdot 2^2 \cdot 0,7071^2 = 26 \text{ kgm}^2 \quad (14)$$

$$I_z = 3 \cdot 2^2 \cdot 3 + \frac{1}{3} \cdot 3 \cdot 4 \cdot 2^2 = 52 \text{ kgm}^2 \quad (15)$$

$$I_{xy} = I_{yx} = m \cdot (-l \cdot \sin 45^\circ) \cdot (-l \cdot \cos 45^\circ) + \int_m x \cdot \sin 45^\circ \cdot x \cdot \cos 45^\circ dm =$$

$$= 3 \cdot (-1,414) \cdot (-1,414) + \frac{m}{l} \cdot 0,7071^2 \cdot \int_0^l x^2 dl = 6 + \frac{4}{2} \cdot 0,5 \cdot \frac{2^3 - 0^3}{3} = 8,666 \text{ kgm}^2 \quad (16)$$

$$[I] = \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix} = \begin{pmatrix} 26 & -8,666 & 0 \\ -8,666 & 26 & 0 \\ 0 & 0 & 52 \end{pmatrix} \quad (17)$$

Invariants A, B, C are calculated using (9), (10) and (11):

$$A = I_x + I_y + I_z = 26 + 26 + 52 = 104 \text{ kgm}^2 \quad (18)$$

$$B = I_x I_y + I_y I_z + I_x I_z - (I_{xz})^2 - (I_{zy})^2 - (I_{yx})^2 =$$

$$= 26 \cdot 26 + 26 \cdot 52 + 26 \cdot 52 - 0 - 0 - (-8,666)^2 = 3304,9 \text{ kgm}^2 \quad (19)$$

$$C = \det \begin{vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{vmatrix} = 26 \cdot 26 \cdot 52 - 52 \cdot 8,666 \cdot 8,666 = 31246,83 \text{ kgm}^2 \quad (20)$$

Equation (8) is composed and solved:

$$I^3 - 104I^2 + 3304,9I - 31246,83 = 0$$

$$I_1 = 17,333 \text{ kgm}^2; I_2 = 34,666 \text{ kgm}^2; I_3 = 52 \text{ kgm}^2 \quad (21)$$

The invariants are verified, noting that the products of inertia for principal axes of inertia are 0:

$$A = I_1 + I_2 + I_3 = 17,333 + 34,666 + 52 = 104 \text{ kgm}^2 \quad (22)$$

$$B = I_1 I_2 + I_2 I_3 + I_1 I_3 - (I_{xz})^2 - (I_{zy})^2 - (I_{yx})^2 =$$

$$= 600,86 + 1802,63 + 901,32 - 0 - 0 - 0 = 3304,8 \text{ kgm}^2 \quad (23)$$

$$C = \det \begin{vmatrix} I_1 & -I_{12} & -I_{13} \\ -I_{21} & I_2 & -I_{23} \\ -I_{31} & -I_{32} & I_3 \end{vmatrix} = \det \begin{vmatrix} 17,333 & 0 & 0 \\ 0 & 34,666 & 0 \\ 0 & 0 & 52 \end{vmatrix} = 31245,02 \text{ kgm}^2 \quad (24)$$

I_1 , I_2 and I_3 (eigenvalues) are the principal mass moments of inertia for the body in fig.1. It is also required to determine the principal axes of inertia. This is done with the directional cosines using (12).

$$\begin{cases} \lambda(26 - I) + \mu(-8,666) + \nu(0) = 0 \\ \lambda(-8,666) + \mu(26 - I) + \nu(0) = 0 \\ \lambda(0) + \mu(0) + \nu(52 - I) = 0 \\ \lambda^2 + \mu^2 + \nu^2 = 1 \end{cases} \quad (25)$$

For $I = I_1 = 17,333 \text{ kgm}^2$ (25) has solutions $\lambda = \pm 0,7071$; $\mu = \pm 0,7071$; $\nu = 0$. Then the unit vector for the axis for which $I_1 = 17,333 \text{ kgm}^2$ is $e_1(0,7071; 0,7071; 0)$.

For $I = I_2 = 34,666 \text{ kgm}^2$ (25) has solutions $\lambda = \mp 0,7071$; $\mu = \pm 0,7071$; $\nu = 0$. Then the unit vector for the axis for which $I_1 = 34,666 \text{ kgm}^2$ is $e_2(-0,7071; 0,7071; 0)$.

For $I = I_3 = 52 \text{ kgm}^2$ (25) has solutions $\lambda = 0$; $\mu = 0$; $\nu = \pm 1$. Then the unit vector for the axis for which $I_3 = 52 \text{ kgm}^2$ is $e_3(0; 0; 1)$.

NB: Here we choose to work with the first sign of the projections of the unit vectors. Eigenvectors are not unique.

The principal axes of inertia are perpendicular to one another and the unit vectors have length of 1, a mathematical verification can be performed:

$$\vec{e}_i \vec{e}_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad (26)$$

The final result for the principal axes of inertia is shown at fig.2:

Figure 2. Principal axes of inertia for the body

3.2. Steps of Analytical Solution:

Appropriate choice of Cartesian coordinate system.

Calculation of the mass moments and products of inertia in that coordinate system – matrix (13).

Determining the eigenvalues and eigenvectors of matrix (13).

The real difficulty actually is the determination of matrix (13). It is pedagogically justified to consider the example given above as an analytical solution. However, it is often times the case that bodies have significantly more complex geometry. Therefore, a numerical solution is to be implemented in practical applications.

3.3. Numerical Implementation

An elegant solution of the considered problem can be obtained using MATLAB. There is a built-in function in it that allows the determination of eigenvalues and eigenvectors of a matrix. The matrix itself is the matrix of the inertial characteristics of the body we are interested in (13). The calculation of the elements of the matrix is based on the geometry of the model. Of course, the model is the one used above, for a comparison of results to be possible. The mass moments of inertia with respect to x, y or z axes are defined using the length of the rods, mass per unit length and point-masses. Comments are added in the MATLAB code for clarity. The code is shown at fig.3. The result of the implementation of the code is shown at fig.4

```

1  %% Program for determining principal mass moments of inertia
2  clear, clc
3  m=3;% Point mass [kg]
4  l=2;% Length of rod [m]
5  ml=2;% Mass per unit length [kg/m]
6  Ix=(1/3)*(1*ml)*(l^2)+m*(l^2)+m*(1*0.7071)^2+(1/3)*(1*ml)*(l^2)*(0.7071)^2;
7  Iy=Ix;% Definition of mass moments of inertia with respect to axes x and y
8  Iz=3*(m*1^2)+(1/3)*3*ml*1*(l^2);% Moment of inertia with respect to z
9  Ixy=ml*0.7071*0.7071*(l^3)/3+m*(1*0.7071)^2;% Product of inertia Ixy
10 Iyx=Ixy;
11 Iyz=0;Izy=Iyz;Izx=Izy;Ixz=Izx;% Products of inertia when one coordinate is 0.
12 A=[Ix -Ixy -Ixz;
13    -Iyx Iy -Iyz;
14    -Izx -Izy Iz];% Matrix of inertial characteristics
15 [ei,Ii]=eig(A)% Determination of eigenvectors and eigenvalues

```

Figure 3. MATLAB code of the eigenvalue problem

```

Command Window

A =
    25.9998    -8.6665         0
   -8.6665    25.9998         0
         0         0    52.0000

ei =
   -0.7071   -0.7071         0
   -0.7071    0.7071         0
         0         0    1.0000

Ii =
    17.3333         0         0
         0    34.6663         0
         0         0    52.0000

fx >>

```

Figure 4. MATLAB Solution of the Problem

As seen from fig.4 and the manual calculation, there is no difference between the two. This is to be expected, as the program code implements the same mathematical ideas. Of course, the rounding of the results is a matter of convention. It is also worth noting that the MATLAB solution takes considerably less time to perform which is an advantage of an engineering point of view. A matrix is formed, that contains the results for the eigenvectors. Its columns are the right eigenvectors, so that $A \cdot e_i = e_i \cdot I_i$ [11]. The diagonal of matrix I_i gives the eigenvalues (principal mass moments of inertia).

4. Conclusions and Recommendations

The problem of determining the principal axes and principal mass moments of inertia is actually an eigenvector and eigenvalue problem. A mathematical apparatus exists that is implemented in various methodologies and programs. The real difficulty occurs when matrix (13) is to be calculated. Of course, most of the shapes applied in engineering practice are well studied. However, sometimes it is required to obtain the numerical values of inertial properties of bodies that are not widely used. In such a case, the process described here can be of some use.

Software products such as Mathcad, MATLAB and Maple etc. provide the means to solve most complex mathematical problems that may arise from any field of science or engineering. What these

products cannot do, however, is to define a problem as such and to describe the physical reality with an appropriate mathematical model. This is done by the scientist or the engineer. And that is why it is expected from the scientist or the engineer to have a deep understanding for the considered physical phenomena.

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